Ternary Quantized Polar Code Decoders: Analysis and Design

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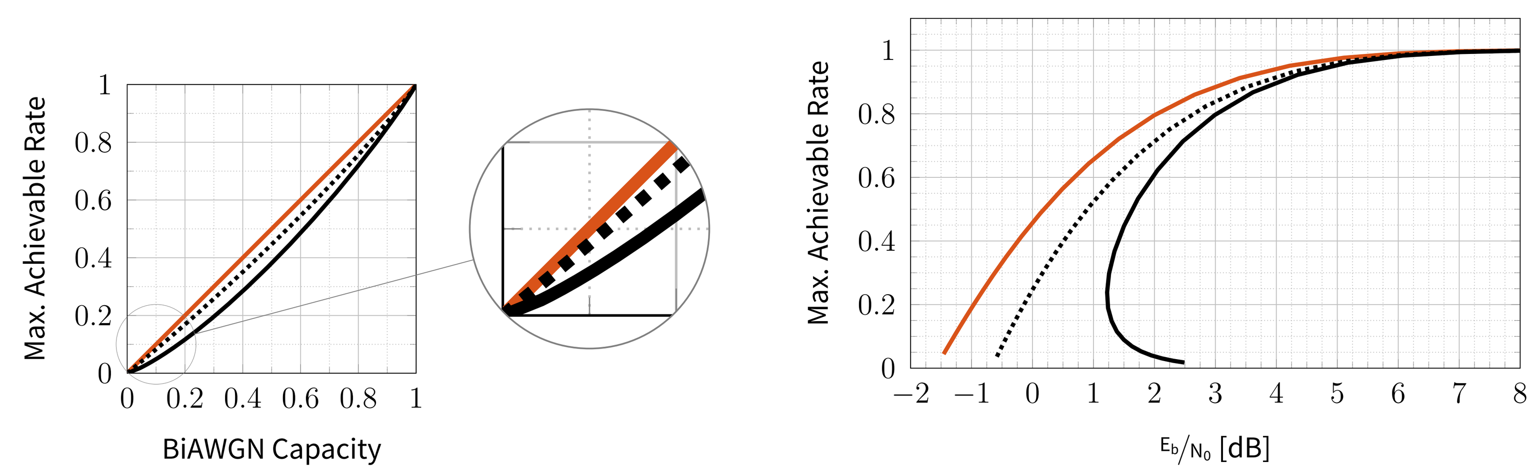
Abstract

What? Analyze and design (list) decoders for polar codes (PC) for 3-level quantized (3Q) channel output, and 3Q log-likelihood ratio (LLR) messages. **Why?** Lower complexity (*e.g.*, IoT) → lower energy consumption & cheaper device production.

Results • Negative impact of coarse quantization underestimated in the literature:



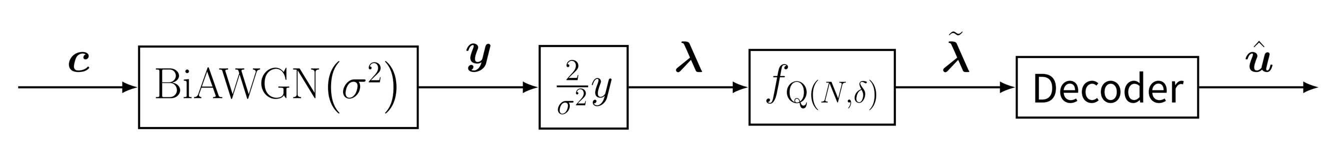
(a) Maximum achievable rate vs. capacity (b) Maximum achievable rate vs. E_b/N_0

Unquantized SC decoding over unquantized BiAWGN — and over 3-level quantized BiAWGN (3Q-BiAWGN) ·····, and 3-level quantized SC decoding over 3Q-BiAWGN —

- LLR quantization → path metric (PM) quantization → impaired list management
- Low-complexity techniques: **In-List ML** and **Expected Path Metric Updates** (EPMU)
- Sizable gains, in particular for low code rates

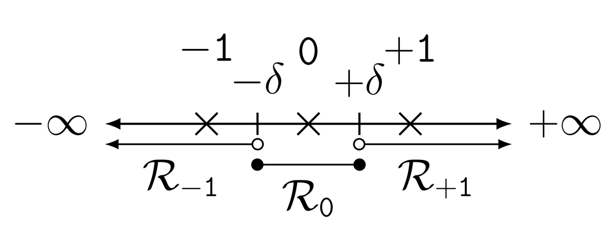
Preliminaries

Three-Level Quantized BiAWGN Channel



Uniform quantization $f_{\mathrm{Q}(3,\delta)}$: $\mathcal{L}_3 \triangleq \{-1, 0, +1\}$

No quantization: $\mathcal{L}_\infty \triangleq \mathbb{R}$



Polar Coding

Encoding:

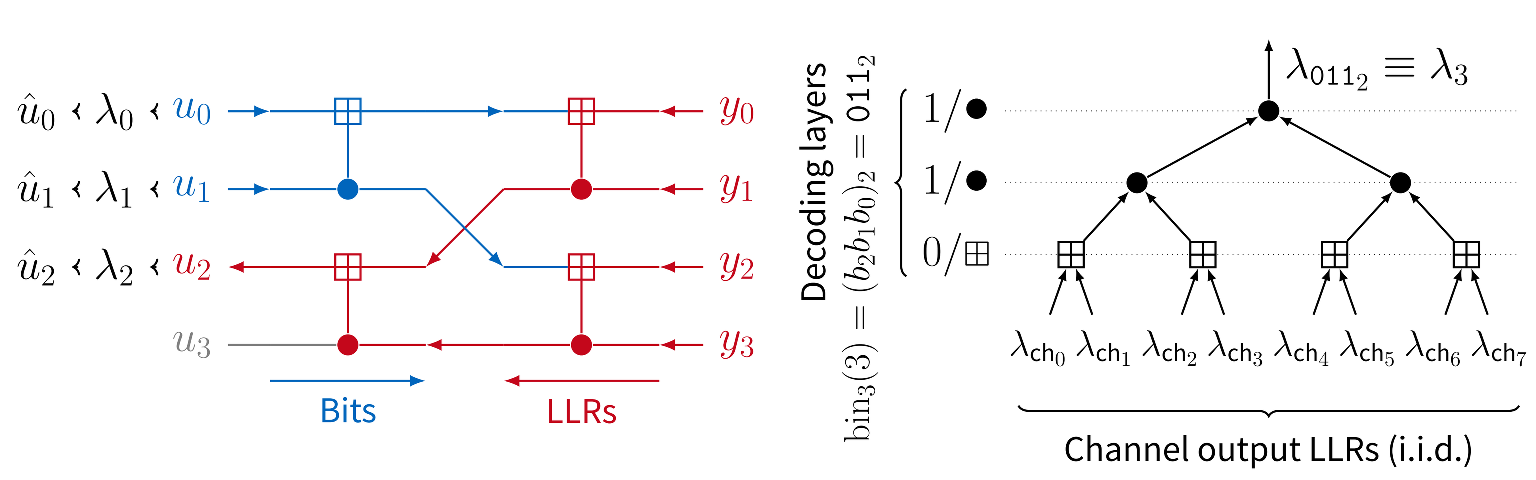
$$\boldsymbol{c} = \boldsymbol{G}_m \boldsymbol{u} \qquad \boldsymbol{G}_m \triangleq \boldsymbol{F}^{\otimes m} \boldsymbol{P}_m^{(\text{bitrev})} \qquad \boldsymbol{F} \triangleq \left[\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right]$$

Synthetic channels:

$$p_{\boldsymbol{Y}\boldsymbol{U}^i|U_i}(\boldsymbol{y}, \boldsymbol{u}^i|u_i) \triangleq \sum_{\boldsymbol{u}_{i+1}^n \in \{0,1\}^{n-i-1}} \frac{1}{2^{n-1}} p_{\boldsymbol{Y}|\boldsymbol{U}}(\boldsymbol{y}|\boldsymbol{u}).$$

Decoding: → LLR message passing over factor graph

$$\lambda_i \triangleq \log\left(\frac{p_{\boldsymbol{Y}\boldsymbol{U}^i|U_i}(\boldsymbol{y}, \hat{\boldsymbol{u}}^i|0)}{p_{\boldsymbol{Y}\boldsymbol{U}^i|U_i}(\boldsymbol{y}, \hat{\boldsymbol{u}}^i|1)}\right) \qquad \lambda_{\text{ch}_i} \triangleq \log\left(\frac{p_{Y_i|C_i}(y_i|0)}{p_{Y_i|C_i}(y_i|1)}\right)$$



VN ●: $x_1 \bullet x_2 \triangleq x_1 + x_2$ CN ⊞: $x_1 \boxplus x_2 \approx \operatorname{sign}(x_1)\operatorname{sign}(x_2)\min\{|x_1|, |x_2|\}$

Path ℓ's metric: $\mathsf{PM}_{\ell,i} = \sum_{j=0}^{i} f_{\mathsf{PM}}(\lambda_{\ell,j}, \hat{u}_{\ell,j})$ $\qquad f_{\mathsf{PM}}(\lambda, u) \approx \max\{0, (-1)^{1-u}\lambda\}$

Key Insights

Unquantized decoder:

- Full-precision LLRs $\lambda \in \mathcal{L}_\infty$
- Full-precision PMs:

  $\mathsf{PM}_\ell = -\log(\Pr[\boldsymbol{U} = \hat{\boldsymbol{u}}_\ell | \boldsymbol{Y} = \boldsymbol{y}])$

  $\approx$ Plausibility of path (ML)

Quantized decoder:

- Distorted LLRs $\tilde{\lambda} \in \mathcal{L}_3$
  - → No magnitude, no reliability info
- Distorted PMs
  - → Bad plausibility measure

PM quantization severely impacts list management!

→ Use **In-List ML** to select most likely codeword among list decoder output!

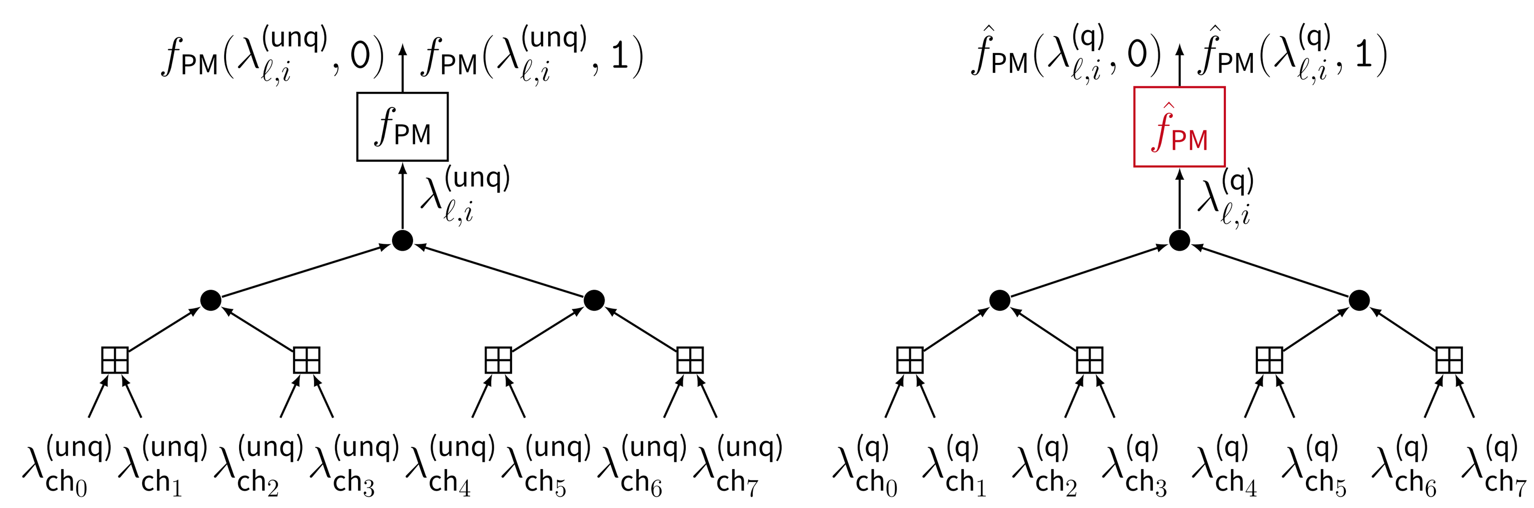
→ Use statistical reliability information for **Expected Path Metric Updates**!

In-List ML

$$\hat{\boldsymbol{c}}_{\mathsf{ML}} = \arg\max_{\boldsymbol{c} \in \mathcal{C}_{\text{list}}} P(\boldsymbol{y}|\boldsymbol{c})$$

Expected Path Metric Updates

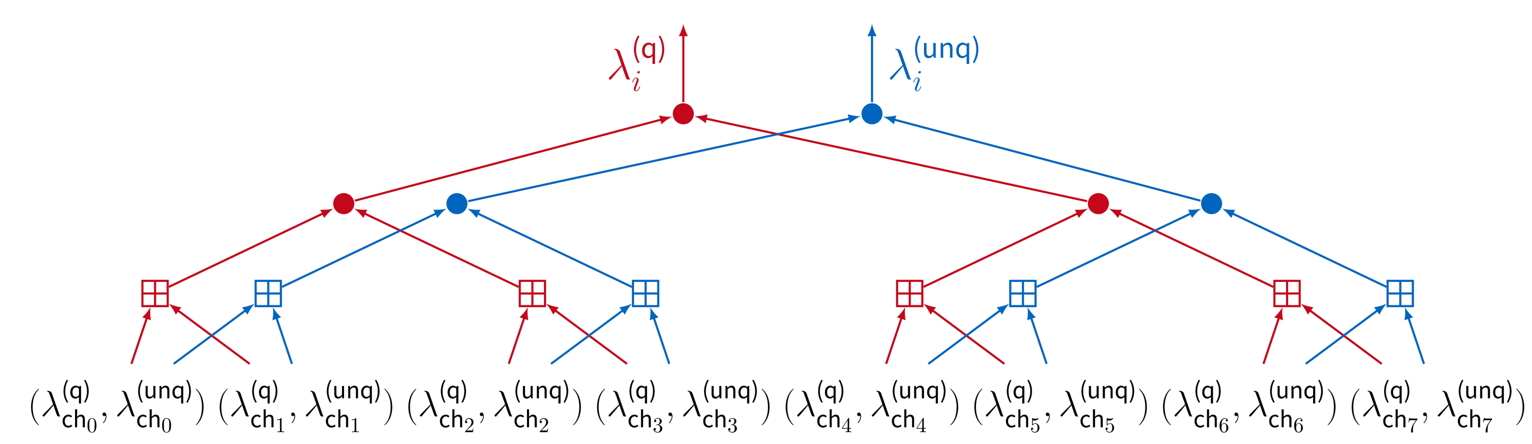
Unquantized decoder: **Quantized decoder:**



If $P\left(\Lambda_i^{(\text{unq})}, \Lambda_i^{(\text{q})}\right)$ was known, for $u \in \{0,1\}$: → MMSE!

$$\min_{\hat{f}_{\mathsf{PM}}} \mathsf{E}\left[\left(f_{\mathsf{PM}}(\Lambda_i^{(\text{unq})}, u) - \hat{f}_{\mathsf{PM}}(\Lambda_i^{(\text{q})}, u)\right)^2\right] \Leftrightarrow \hat{f}_{\mathsf{PM}}(\lambda_i^{(\text{q})}, u) \triangleq \mathsf{E}\left[f_{\mathsf{PM}}(\Lambda_i^{(\text{unq})}, u) \,\middle|\, \Lambda_i^{(\text{q})} = \lambda_i^{(\text{q})}\right]$$

How to obtain $P\left(\Lambda_i^{(\text{unq})}, \Lambda_i^{(\text{q})}\right)$? → Joint density evolution ('$\mathcal{L}_{(3,\infty)}$-SC decoder')!

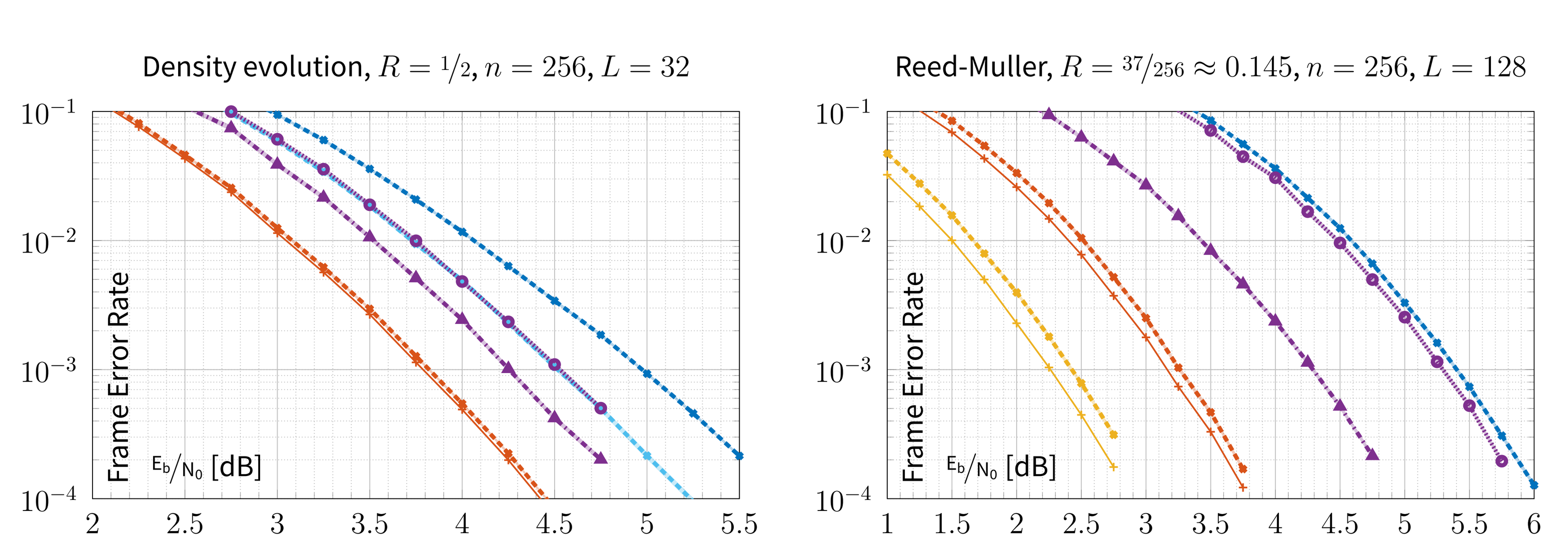


For $x_1, x_2 \in \mathcal{L}_{(3,\infty)} \triangleq \mathcal{L}_3 \times \mathcal{L}_\infty$:

$$x_1 \boxplus x_2 = \left(x_1^{(\text{q})}, x_1^{(\text{unq})}\right) \boxplus \left(x_2^{(\text{q})}, x_2^{(\text{unq})}\right) \triangleq \left(x_1^{(\text{q})} \boxplus x_2^{(\text{q})}, x_1^{(\text{unq})} \boxplus x_2^{(\text{unq})}\right)$$

$$x_1 \bullet x_2 = \left(x_1^{(\text{q})}, x_1^{(\text{unq})}\right) \bullet \left(x_2^{(\text{q})}, x_2^{(\text{unq})}\right) \triangleq \left(x_1^{(\text{q})} \bullet x_2^{(\text{q})}, x_1^{(\text{unq})} \bullet x_2^{(\text{unq})}\right)$$

Simulation Results



(a) Gains for In-List ML and EPMU (b) Utility of EPMU at low code rates

Channel	Decoder	Metric
3Q-BiAWGN	$\mathcal{L}_3$-SCL	PM-FER
3Q-BiAWGN	$\mathcal{L}_3$-SCL	List-FER
3Q-BiAWGN	$\mathcal{L}_3$-SCL + In-List ML	LML-FER
3Q-BiAWGN	$\mathcal{L}_3$-SCL + In-List ML + EPMU	LML-FER

Channel	Decoder	Metric
3Q-BiAWGN	$\mathcal{L}_\infty$-SCL	PM-FER
3Q-BiAWGN	$\mathcal{L}_\infty$-SCL	ML-LB
BiAWGN	$\mathcal{L}_\infty$-SCL	PM-FER
BiAWGN	$\mathcal{L}_\infty$-SCL	ML-LB

FER vs. E_b/N_0 for (a) $R = 1/2$ Polar code and (b) $R = 37/256$ Reed-Muller code

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[2] Joachim Neu. "Quantized Polar Code Decoders: Analysis and Design". arXiv:1902.10395v1 [cs.IT]. MA thesis. Technical University of Munich, 2018.